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Logic of Infinite Quantum Systems

Daniele Mundici[1]

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Limits of sequences of finite-dimensional (AF) C^*-algebras, such as the CAR algebra for the ideal Fermi gas, are a standard mathematical tool to describe quantum statistical systems arising as thermodynamic limits of finite spin systems. Only in the infinite-volume limit one can, for instance, describe phase transitions as singularities in the thermodynamic potentials, and handle the proliferation of physically inequivalent Hilbert space representations of a system with infinitely many degrees of freedom. As is well known, commutative AF C^*-algebras correspond to countable Boolean algebras, i.e., algebras of propositions in the classical two-valued calculus. We investigate the *noncommutative* logic properties of general AF C^*-algebras, and their corresponding systems. We stress the interplay between Gödel incompleteness and quotient structures—in the light of the "nature does not have ideals" program, stating that there are no quotient structures in physics. We interpret AF C^*-algebras as algebras of the infinite-valued calculus of Lukasiewicz, i.e., algebras of propositions in Ulam's "*twenty questions*" game with lies.

INTRODUCTION

In the quantum theory of finite systems, observables are represented by self-adjoint operators in a certain Hilbert space H. The pure states of the system are the extremal positive linear normalized functionals on the C^*-algebra $B(H)$ of bounded linear operators on H. The dynamics is specified by a one-parameter group of automorphisms of $B(H)$. States are weighted sums of pure states, and unbounded observables are approximated by bounded ones. By von Neumann's uniqueness theorem, the coordinates, momenta, and spins of an arbitrary finite system admit precisely one irreducible Hilbert space representation, up to unitary equivalence.

According to the theory of Birkhoff and von Neumann (1936), propositions expressing properties of a physical system are represented by projections on $B(H)$; the set of propositions is naturally equipped with the inf and

[1]Department of Computer Science, University of Milan, I-20135 Milan, Italy.

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sup operations, and with a form of complementation that are reminiscent of the Boolean connectives of conjunction, disjunction, and negation.

While the Hilbert space approach scored many successes for systems having finitely many degrees of freedom, it was grudgingly recognized that infinite systems are beyond the scope of this formalism (Emch, 1984, p. 361; Sewell, 1986). One can no longer speak of *the* Hilbert space of the system. In fact, the observables of an infinite system usually have a host of physically inequivalent representations, corresponding to macroscopically different classes of states. An essential feature of macroscopic assemblies of particles is that the state equations are size independent. We are naturally led to an idealization of the macroscopic system as an infinite-volume limit of increasingly large finite systems with constant density. In this way one can, for instance, describe phase transitions as singularities in the thermodynamic potentials.

A mathematical counterpart of this construction is provided by approximately finite-dimensional (AF) C^*-algebras. By definition, an *AF C^*-algebra* is the norm closure of the union of an ascending sequence of finite-dimensional C^*-algebras, all with the same unit. Introduced in successive stages by Glimm, Dixmier, and Bratteli (see Bratteli, 1972), AF C^*-algebras are now the standard tool for the algebraization of spin systems and the like (Sewell, 1986; Bratteli and Robinson, 1979). For example, the ideal Fermi gas is described by the *CAR algebra*—the limit C^*-algebra $\mathrm{B}(\mathbf{C}) \subset \mathrm{B}(\mathbf{C}^2) \subset \mathrm{B}(\mathbf{C}^4) \subset \mathrm{B}(\mathbf{C}^8) \subset \cdots$. This description is free of any underlying Hilbert space structure: all the information is contained in an AF C^*-algebra (Emch, 1984, pp. 362, 456).

It follows that all the logical machinery of systems described by AF C^*-algebras is necessarily built in the algebra itself, independently of any Hilbert space representation.

In this paper we shall interpret projections of AF C^*-algebras as propositions in the infinite-valued sentential calculus of Lukasiewicz (Tarski and Lukasiewicz, 1956). We then apply to AF C^*-algebras such notions as polynomial time computability and Gödel incompleteness. We also relate states of AF C^*-algebras and truth-averaging processes.

We refer to Dixmier (1977) for background on C^*-algebras, and to Effros (1981) and Goodearl (1982) for AF C^*-algebras. Throughout this paper, each C^*-algebra A has a unit 1_A and is separable.

1. ELLIOTT'S CLASSIFICATION AND MV ALGEBRAS

It is not hard to see that every commutative AF C^*-algebra is isomorphic to the C^*-algebra $\mathrm{C}(X)$ of all continuous complex-valued functions defined over a separable Boolean (totally disconnected, compact Hausdorff) space X. This suggests that AF C^*-algebras should be regarded as sort of noncommutative Boolean algebras (Blackadar, 1987, 7.1). To substantiate this intuition, we rely on Elliott's (1976) classification theory.

Recall that two projections p and q in a C^*-algebra A are *equivalent* (in the sense of Murray–von Neumann) iff there is an element $v \in A$ such that $vv^* = p$ and $v^*v = q$. The set of equivalence classes is denoted by $D(A)$, and the equivalence class of p is denoted by $[p]$. For AF C^*-algebras this notion of equivalence coincides with the familiar notion of unitary equivalence (Effros and Rosenberg, 1978, 3.6; Blackadar, 1987, 7.1). When A is the C^*-algebra $B(\mathbf{C}^n)$ of linear operators on n-dimensional Hilbert space, p is equivalent to q iff dim(range p) = dim(range q). Accordingly, equivalence classes of projections are often regarded as (generalized) dimensions.

In its original formulation, Elliott's (1976) classification theory is the study of the order-theoretic and additive properties of dimensions.

We write $[p] \leq [q]$ iff p is equivalent to a subprojection of q. In many important cases, including all AF C^*-algebras, the relation $\leq$ on $D(A)$ is a partial order (i.e., a reflexive, transitive, antisymmetric binary relation), and is called the *Murray–von Neumann order* of A. A partial addition $+$ is defined on $D(A)$ by stipulating that $[p] + [q]$ exists iff p and q are, respectively, equivalent to orthogonal projections p' and q'. Defining in this case $[p] + [q] = [p' + q']$, $D(A)$ becomes a partial structure, known as *Elliott's local semigroup.* Trivially, Elliott's addition is associative, commutative, monotone, and satisfies the following residuation property, where for each projection $p \in A$, $[p]^*$ is an abbreviation of $[1_A - p]$:

$[p]^*$ is the smallest element in D whose sum with $[p]$ equals $[1_A]$.

The requirement that property (*) be preserved together with associativity, commutativity, and monotonicity makes the extension problem for Elliott's addition a nontrivial and interesting one:

Theorem 1. (Mundici and Panti, n.d.). For any AF C^*-algebra A, let $D = D(A)$. Then:

(i) Elliott's addition $+$ has at most one extension to an associative, commutative, monotone operation $\oplus: D^2 \to D$ satisfying Condition (*).

(ii) The (unique) extension $\oplus$ exists if, and only if, the Murray–von Neumann order of A is a lattice-order.

In his classification theory of AF C^*-algebras, Elliott (1976) proved that the local semigroup $(D(A), +)$ is a complete invariant for each AF C^*-algebra A. It follows that:

Theorem 2. The semigroup $(D(A), \oplus)$ is a complete invariant for all AF C^*-algebras A whose Murray–von Neumann order is lattice: nonisomorphic AF C^*-algebras determine nonisomorphic semigroups.

For all $x, y \in D(A)$, the $\oplus$ operation can express the order as follows: $x \leq y$ iff $x \oplus z = y$ for some $z \in D(A)$. Let 0 and 1, respectively, denote the smallest and the largest element of D. From Condition (*) it follows that the semigroup $(D(A), \oplus)$ is equipped with a unary operation *, where x^* is the smallest element y such that $x \oplus y = 1$. Setting now $x \cdot y = (x^* \oplus y^*)^*$, we obtain a map $A \to (D(A), 0, 1, {}^*, \oplus, \cdot)$.

We shall now characterize the range of this map. Following Chang (1958), we say $B = (B, 0, 1, {}^*, \oplus, \cdot)$ is an *MV algebra* iff B satisfies the following equations:

MV1	$(x \oplus y) \oplus z = x \oplus (y \oplus z)$
MV2	$x \oplus y = y \oplus x$
MV3	$x \oplus 0 = x$
MV4	$x \oplus 1 = 1$
MV5	$0^* = 1$
MV6	$1^* = 0$
MV7	$x \cdot y = (x^* \oplus y^*)^*$
MV8	$(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x.$

Replacing y by 0 in the last equation, we get $x^{**} = x$. Replacing y by 1, we get $1 = x^* \oplus x$. Then it is not hard to see (Mundici, 1986, §2) that these equations are equivalent to Chang's original equations. Boolean algebras coincide with MV algebras obeying $x \oplus x = x$. The prototypical example of an MV algebra is the unit interval [0, 1] equipped with the operations $x^* = 1 - x$, $x \oplus y = \min(1, x + y)$, $x \cdot y = \max(0, x + y - 1)$. Indeed, *Chang's* (1959) *completeness theorem* states that the variety of MV algebras coincides with the smallest class of structures containing [0, 1] and closed under homomorphic images, subalgebras, and products. Stated otherwise, if an equation holds in [0, 1], then it holds in every MV algebra. Subalgebras of [0, 1] exhaust all possible cases of *simple* (i.e., quotient-free) MV algebras. Algebras of [0, 1]-valued functions exhaust the most general possible case of *semisimple* MV algebras, those algebras where the intersection of maximal ideals is zero (Chang, 1959; Belluce, 1986).

MV algebras provide the desired notion of noncommutative Boolean algebra in the following sense:

Theorem 3. (Mundici and Panti, n.d.; Mundici, 1986). Up to isomorphism, the map $A \to (D(A), 0, 1, {}^*, \oplus, \cdot)$ is a one–one correspondence between AF C^*-algebras whose Murray–von Neumann order is a lattice and countable MV algebras. Commutative AF C^*-algebras correspond to countable Boolean algebras.

Some instances of the above duality theorem are given by Table I.

Table I.

Countable MV algebra	Its AF C^*-correspondent
$\{0, 1\}$	**C**
Lukasiewicz chain $\{0, 1/n, \ldots, (n-1)/n, 1\}$	$B(\mathbf{C}^n)$, the $n \times n$ complex matrices
Finite	Finite-dimensional
Boolean	Commutative
Atomless Boolean	$C(2^\omega)$ (Effros, 1981, p. 13)
Totally ordered	With Murray–von Neumann comparability of projections (Elliott, 1979)
Subabgebra of $\mathbf{Q} \cap [0, 1]$	Glimm's UHF algebra (Effros, p. 50; Effros and Rosenberg, 1978)
Dyadic rationals in [0, 1]	CAR algebra of the Fermi gas (Blackadar, 1987; Effros, 1981)
$\mathbf{Q} \cap [0, 1]$	Glimm's universal UHF algebra (Effros, 1981; Effros and Rosenberg, 1978)
Subalgebra of [0, 1]	Simple with Murray–von Neumann comparability (Elliott, 1979)
Generated by an irrational $\rho \in [0, 1]$	Effros–Shen algebra $\mathfrak{F}_\rho$ (Effros, 1981, p. 65)
All real algebraic numbers in [0, 1]	Blackadar algebra B (Blackadar, 1980*a*, p. 504)
Two-ideal algebras of Mundici (1992*a*)	Behncke–Leptin algebras with two-point dual (Behncke and Leptin, 1972)
Chang algebra C (Chang, 1958, p. 474)	Behncke–Leptin algebra $A_{0,1}$ (Behncke and Leptin, 1972; Bratteli, 1972, 3.4)
Every prime quotient is finite	Liminary with Hausdorff spectrum (Cignoli *et al.*, n.d.)
Finite-valued (Grigolia, 1977)	Subhomogeneous with Hausdorff spectrum (Mundici, n.d.-*a*)
Three-valued (Grigolia, 1977)	3-Subhomogeneous with Hausdorff spectrum (Mundici, 1989)
Post MV algebra of order $n + 1$ (Mundici, 1993)	Homogeneous of order n (Mundici, n.d.-*a*)
Finite product of Post MV algebras (Mundici, 1993)	Continuous trace (Mundici, n.d.-*a*)
Free, denumerably many generators (Mundici, 1986, §4)	Universal AF C^*-algebra $\mathfrak{M}$ (Mundici, 1986, §8)
Free, one generator	The "Farey" AF C^*-algebra $\mathfrak{M}_1$ (Mundici, 1988a)

2. NONCOMMUTATIVE (MANY-VALUED) LOGIC AND ULAM'S GAME

MV algebras were originally introduced as the algebras of the infinite-valued sentential calculus of Lukasiewicz (Tarski and Lukasiewicz, 1956). To investigate the "noncommutative" logical aspects of MV algebras, we consider Ulam's game (Ulam, 1976, p. 281), a variant of the "*twenty questions*" game where the first player thinks of a number x in a certain set

S, and the second player must guess x by asking questions to which the first player can only answer yes or no—being allowed to lie a certain number L of times. Let us call Pinocchio the first player.

In practical applications, Pinocchio may be replaced by a satellite, and Pinocchio's answers are bits of information transmitted by the satellite from a great distance. Distortion during transmission has the same effect as Pinocchio's lies. The role of the second player is now taken by a powerful receiver-transmitter sending noiseless feedback information to the satellite, in order to optimize the error-correction process. Typically, the feedback may consist in sending a copy of the received bit back to the satellite. In this way, Ulam's game becomes an interesting chapter of the theory of error-correcting communication with feedback (Berlekamp, 1968; Czyzowicz *et al.*, 1989; Mundici, 1991).

When $L > 0$, Pinocchio's answers do not obey classical logic, in at least two respects: (i) the conjunction of two equal answers to the same repeated question is usually more informative than a single answer, and (ii) the conjunction of two opposite answers need not lead to contradiction.

During the game, our state of knowledge about Pinocchio's secret number x becomes sharper and sharper; an increasing set of numbers is excluded from consideration, as soon as they falsify too many (i.e., $\geq L + 1$) answers. Needless to say, after receiving t answers $A_1, \ldots, A_t$, our state of knowledge k is only determined by the conjunction of these answers. Without loss of generality, k is representable by a function $k: S \to \{0, 1/(L + 1), \ldots, L/(L + 1), 1\}$, where for every $y \in S$, $k(y)$ is the distance, measured in units of $L + 1$, from the condition of falsifying too many answers. Thus, in particular, $k(y) = 1 - w/(L + 1)$ iff y falsifies w answers, $w = 0, 1, \ldots, L + 1$. As proved in Mundici (1992*b*), upon identifying A_i with the state of knowledge determined by the ith answer alone, we have identity $k = A_1 \cdot \cdots \cdot A_t$, where $\cdot$ is the Lukasiewicz conjunction $a \cdot b = \max(0, a + b - 1)$.

To express the natural pointwise order between states of knowledge $k \leq h$ (saying that k is more restrictive, or sharper, than h), we can use the negation operation $h^* = 1 - h$. Indeed, the inequality $k \leq h$ holds iff $k \cdot h^* = 0$. In this way, inequalities can be equivalently reformulated as equations.

Let $K_{L,S}$ be the algebra of states of knowledge equipped with the operations $\cdot$, $*$, and $\oplus$, where $\oplus$ is the de Morgan dual of conjunction, $a \oplus b = (a^* \cdot b^*)^*$. Let us agree to say that an equation is *absolute* iff it holds in $K_{L,S}$ for all finite L and S. Associativity and commutativity are examples of absolute equations, while the law of idempotence $h \cdot h = h$ is not absolute, as it holds only for states of knowledge in Ulam's game without lies. Then we have:

Theorem 4. (Mundici, 1993). An equation is absolute iff it follows from the equations MV1–MV8 iff it holds in the MV algebra $\mathbf{Q}\cap[0, 1]$ of rational numbers in the unit interval.

As a consequence, we may safely identify the following: (i) tautologies, or equivalences, in the infinite-valued sentential calculus of Lukasiewicz, (ii) equations which follow from the MV equations MV1–MV8, (iii) equations which are valid when interpreted over $\mathbf{Q}\cap[0, 1]$, and (iv) equations $k = h$ between states of knowledge in Ulam's game which are valid even if we do not know how many times Pinocchio can lie.

Boolean logic corresponds to the special case of Ulam's game where Pinocchio is not allowed to lie. Thus, it is suggestive to regard the noncommutativity of MV algebras as a logical counterpart of the lies (or distortions) in Ulam's game.

As a matter of fact, one can naturally extend Ulam games from finite to infinite search spaces, S, and even assume that the number of lies available to Pinocchio is not a constant L, but is a variable $l(x)$ depending on the chosen number x. In this way, larger and larger classes of MV algebras arise as algebras of states of knowledge in suitably generalized Ulam games.

Conversely, Di Nola's (n.d.) representation theorem yields a fixed nonstandard real line $\mathbf{R}^*$ together with its unit interval $[0, 1]^* = \{z \in \mathbf{R}^* | 0 \le z \le 1\}$ such that every countable MV algebra B is an algebra of $[0, 1]^*$-valued functions defined over some space S. This allows us to realize B as an algebra of states of knowledge in some Ulam game with lies: each state of knowledge $k \in B$ assigns to every $y \in S$ a (possibly nonstandard) truth value $k(y) \in [0, 1]^*$; again, the truth value is interpreted as the relative distance of y from the condition of elimination. Semisimplicity of B is equivalent to requiring that each state of knowledge of B is a $[0, 1]$-valued function.

In this way, C^*-algebraic notions can be given a game-theoretic interpretation, as in Table II [see Mundici (1993) for further details and particular cases].

3. EASY COMPUTATIONS AND GÖDEL INCOMPLETENESS IN AF C^*-ALGEBRAS

After all, equations between states of knowledge in Ulam's game are formulas in the infinite-valued calculus of Lukasiewicz. Formulas are just strings of symbols built up in the usual way from a denumerably infinite supply of sentential variables $X_1, X_2, \ldots$, and the connectives of negation $*$, disjunction $\oplus$, and conjunction $\cdot$.

Identifying two formulas whenever they are logically equivalent, we obtain the *free MV algebra* F_ω over countably many free generators. More

Table II.

Ulam game	AF C^*-algebra
Game	AF C^*-algebra A
Search space S	Space prim A of primitive ideals of A
Element $x \in S$	Kernel of irreducible representation π
Number of lies $l(x)$	$-1 + \dim \pi$
Arbitrary question	Arbitrary subset of prim A
Opposite question	Complementary subset
Finite search space	Finite-dimensional C^*-algebra
Fixed number L of lies	A is homogeneous of degree $L+1$
No lies	Commutative
State of knowledge	Equivalence class of projections
Comparison of states	Murray–von Neumann order
Initial state	Unit element
Inconsistent state	Zero element
Mutually incompatible states	Orthogonal projections
Largest state incompatible with another state	Complementary projection
Conjunction	De Morgan dual of canonical extension of Elliott's addition
Final state of knowledge	Projection which is nonzero in just one irreducible representation

generally, for every set Θ of formulas, identifying two formulas whenever they are equivalent in Θ, we obtain the *Lindenbaum algebra* B_Θ of Θ. Stated otherwise, $B_\Theta = F_\omega / I_\Theta$, where I_Θ is the ideal of F_ω canonically determined by (the negations of the formulas of) Θ.

Algebraists use to say that Θ is a *presentation* of B_Θ via generators (the sentential variables) and relations (the axioms of Θ).The *word problem* for B_Θ is the problem of deciding whether an arbitrary formula ψ is a *consequence* of Θ—i.e., whether ψ follows via modus ponens from the axioms of Θ together with the tautologies of the infinite-valued calculus (Tarski and Lukasiewicz, 1956; Mundici, 1986, §5). Mundici (1988*c*) is devoted to the study of consequence relations in the infinite-valued calculus of Lukasiewicz.

Since Θ uniquely determines B_Θ, and B_Θ uniquely corresponds to an AF C^*-algebra A_Θ, regarding Θ as a presentation of A_Θ we may measure the complexity of A_Θ in terms of the complexity of the decision problem of Θ.

Conversely, for every AF C^*-algebra A whose Murray–von Neumann order is a lattice, we may ask for a simplest possible set of formulas Θ such that $A = A_\Theta$.

Table III summarizes the complexity of many AF C^*-algebras existing in the literature; for more details, and for background in complexity theory, see the references mentioned.

Table III.

AF C^*-algebras	Complexity of simplest presentation Θ
Finite-dimensional	Polynomial time (Mundici, 1987*a*)
Glimm's universal UHF algebra	Polynomial time (Mundici, 1987*a*)
CAR algebra	Polynomial time (Mundici, 1987*a*)
Most Glimm's UHF algebras	Not recursively enumerable
Universal AF C^*-algebra $\mathfrak{M}$	Co-NP complete (Mundici, 1987*a*,*b*)
AF C^* algebra $\mathfrak{M}_1$	Polynomial time
Effros–Shen algebra $\mathfrak{F}_\theta$, θ a quadratic irrational	Polynomial time (Mundici, 1987*a*)
Effros–Shen algebra $\mathfrak{F}_{1/e}$	Polynomial time (Mundici, 1987*a*)
Most Effros–Shen algebras	Not recursively enumerable
Behncke–Leptin algebras with two-point dual	Polynomial time (Mundici, 1992*b*)
Most AF C^*-algebras	Not recursively enumerable

See Mundici (1986) for an explicitly given presentation for the CAR algebra. From the proof of the main result of Mundici (1992*a*) one can draw a concrete presentation of every Behncke-Leptin C^*-algebra with a two-point dual. Once an infinite physical system is described by an AF C^*-algebra A, and A is presented as A_Θ for some set Θ of formulas, the word problem, of A_Θ is the problem of deciding the validity of arbitrary $(*, \oplus, \cdot)$-equations between equivalence classes of projections of A_Θ. Using the expressive power of these equations, one can formalize comparisons, as well as commutativity, incompatibility, and orthogonality properties of the $\{0, 1\}$-valued observables of the system. Moreover, since linear combinations of projections are dense in A, computational problems involving more general observables of the system can also be formalized—or approximated—in terms of the decision problem of Θ.

At the opposite extreme of polynomial time computability, we say that a set of formulas Θ is *Gödel incomplete* iff:

(i) Θ is effectively presentable, in the sense that there is a Turing machine listing all the formulas of Θ, and hence, by Chang's completeness theorem, all consequences of Θ.

(ii) Θ is undecidable, in the sense that there is no Turing machine deciding whether or not a formula ψ is a consequence of Θ.

In Mundici (1986, 6.1) the following theorem is proved:

Theorem 5. Suppose Θ is Gödel incomplete. Then A_Θ has some nontrivial quotient.

The theorem shows that writing $A = A_\Theta$ is more than a notational expedient, because the degree of complexity of Θ impinges upon the ideal structure of A_Θ. Let us, for instance, consider the belief that "nature does not have ideals" (Cuntz, 1982, p. 85), in the sense that the C^*-algebras of physically interesting systems do not have nontrivial quotient structures (Kastler, 1982, p. 468). Accordingly, whenever a C^*-algebra A possesses a nontrivial ideal, the relevant object is the quotient, rather than A (Haag and Kastler, 1964, p. 852); iterating this quotient-elimination procedure, one finally obtains a simple (i.e., quotient-free) structure $A' = A/K'$, where K' is a maximal closed two-sided ideal of A. Writing $A = A_\Theta$ and recalling that all quotient structures are preserved in the duality between AF C^*-algebras and MV algebras (Effros, 1981; Goodearl, 1982; Blackadar, 1987), it follows that K' canonically determines a set of sentences $\Theta' \supseteq \Theta$ such that $A' = A_{\Theta'}$, and Θ' is maximally consistent (i.e., Θ' is consistent, but adding any new formula to Θ' destroys consistency). Now, if Θ is assumed to be *essentially incomplete* (whence Θ is Gödel incomplete, and none of its maximally consistent extensions is recursively enumerable), then by the above theorem, A_Θ has a nontrivial ideal; however, any completion process $\Theta' \supseteq \Theta$ paralleling the quotient elimination process $A \to A'$ will inevitably destroy the effective presentability of A_Θ.

Thus, in some cases one has to strike a balance between the two natural but conflicting desiderata of simplicity and effective presentability.

4. STATES ON MV ALGEBRAS AND ON AF C^*-ALGEBRAS

Intuitively, C^*-algebraic states are averaging processes for the values of observables; MV states are averaging processes for truth-values. Their relationship is discussed in the present section.

By a *state* on an MV algebra B we mean a function $s: B \to [0, 1]$ such that $s(0) = 0$, $s(1) = 1$, and whenever $a \cdot b = 0$, then $s(a \oplus b) = s(a) + s(b)$. We say that s is *faithful* iff $a \neq 0$ implies $s(a) > 0$. We define *invariance* of s with respect to the group of all automorphisms of B in the obvious sense.

Recall that a *tracial state* on a C^*-algebra A is a normalized positive linear functional t satisfying $t(aa^*) = t(a^*a)$ for all $a \in A$. We say that t is *faithful* iff $0 < a \in A$ implies $t(a) > 0$.

Theorem 6. (Mundici, n.d.-*b*). Let A be an AF C^*-algebra whose Murray–von Neumann order is a lattice. Let B be its corresponding MV algebra. Then the tracial states of A are in one–one correspondence with the states of B. Faithful tracial states of A correspond to faithful states of B.

The set of states of an MV algebra B has a convex structure, and inherits the topology of the product space $[0, 1]^B$ of all $[0, 1]$-valued functions over B. Recall from Belluce (1986) that the *spectral topology* of the set prim B of prime ideals of B has as its open sets the sets O_J, where J ranges over the ideals of B, and $O_J = \{p \in \text{prim } B | p \text{ does not contain } J\}$.

Theorem 7. (Mundici, n.d.-*b*). Let B be an MV algebra. It follows that:

(i) Every state of B is in the closure of the convex hull of the set of extremal states of B. The latter form a nonempty compact Hausdorff space which is canonically homeomorphic to the space of maximal ideals of B with the spectral topology.

(ii) B has exactly one state iff B is *local* (i.e., B has only one maximal ideal).

(iii) If B is simple, then the only state of B is the embedding into $[0, 1]$.

(iv) If B is free, then B has an invariant rational-valued faithful state.

(v) If B is semisimple and countable, then B has a faithful state.

(iv) If B is not semisimple, then B has no faithful state.

Following Christensen (1982) and Maeda (1990, and references therein), we say, that a *measure* on the set proj(A) of projections of a C^*-algebra A is a function $m: \text{proj}(A) \to [0, 1]$ such that $m(0) = 0$, $m(1) = 1$, and $m(p) + m(q) = m(p + q)$ whenever $pq = 0$. We say that m is *faithful* iff $p > 0$ implies $m(p) > 0$. We say that m is *invariant* iff $m(p) = m(\alpha(p))$ for every projection p and every automorphism α of A.

By Gelfand's theorem, every commutative C^*-algebra A is isomorphic to the C^*-algebra C(X) for some compact Hausdorff space X. In this case, projections are the (characteristic functions of) clopens of X, and a measure is simply a *finitely additive* measure on the clopens of X.

When projections do not abound, one usually considers the σ-algebra of Borel subsets of X, replacing finite additivity by σ-additivity. Mackey asked whether every σ-additive measure of the set of projections of A is the restriction of a positive linear functional over A. In a celebrated paper, Gleason gave a positive answer for the case when $A = \text{B}(H)$, and the Hilbert space H is of dimension at least three. Various people have studied generalizations of Gleason's theorem. See Maeda (1990) for further information.

At the other extreme, since for every AF C^*-algebra A the linear span of projections is dense in A, there is no need to resort to infinitary operations. We then have:

Theorem 8. (Mundici, n.d.-*b*). Let A be an AF C^*-algebra whose Murray–von Neumann order is a lattice. Let B be the corresponding MV algebra. Then invariant measures on the projections of A are in one–one correspondence with invariant states on B. Faithful invariant measures correspond to faithful invariant states.

5. CONCLUDING REMARKS

We have formalized AF C^*-algebraic systems in terms of sets of formulas in the infinite-valued sentential calculus of Lukasiewicz. We have then applied to AF C^*-algebras methodologies arising from logic and computation theory. For instance, as an effect of (noncommutative) Gödel essential incompleteness, we have seen that effective presentability may irremediably conflict with the traditional desideratum that a physically relevant structure does not have quotients.

At the moment of writing this paper, no example is known of an essentially incomplete AF C^*-algebra arising from a natural quantum system. However, since for many years it was widely—and wrongly—believed that undecidable sentences would never seriously affect the working mathematician, the prospect that Gödel incompleteness phenomena will affect the working mathematical physicist should not be hastily excluded.

New interesting problems arise, whose formulation would have been impossible before this formalization, e.g., the problem of giving algebraic characterizations of AF C^*-algebras having decidable (resp., polynomial time computable, Gödel incomplete, essentially Gödel incomplete) presentations.

MV algebras are known under several different names, and as such, they have been studied by quite a few researchers working in different areas of mathematics, other than many-valued logic. See Cignoli *et al.* (1994) for a comprehensive account on MV algebras.

For instance, as proved in Mundici (1986, §3), MV algebras are categorically equivalent to lattice-ordered Abelian groups with a distinguished strong unit. Indeed, up to isomorphism, Grothendieck's functor K_0 maps AF C^*-algebras whose Murray–von Neumann order is a lattice one–one onto countable Abelian lattice groups with strong unit (Mundici, 1986).

More generally, K_0 maps AF C^*-algebras one–one onto countable unperforated Abelian Riesz groups with strong unit (Effros, 1981; Goodearl, 1982). For the relationship between these groups and lattice-ordered groups, see Mundici (1986), Mundici and Panti (n.d.), and Elliott and Mundici (1993).

By the main theorem of Mundici (1988*a*), every AF C^*-algebra A whose Murray–von Neumann order is a lattice has the *ultrasimplicial* property (Elliott, 1979; Blackadar, 1987, 7.7.2) in the sense that the approximating sequence for A of finite-dimensional C^*-algebras $F_1 \subset F_2 \subset F_3 \subset \cdots$ can always be assumed to correspond, via K_0, to an increasing sequence of free Abelian groups of finite rank with the product order, $\mathbf{Z}^{n_1} \subset \mathbf{Z}^{n_2} \subset \mathbf{Z}^{n_3} \subset \cdots$. This is not true in general for AF C^*-alge-

Table IV.

Finite systems	Infinite systems
Hilbert space H	AF C^*-algebra A
Projections	Equivalence classes of projections (dimensions)
Lattice operations	Additive operations
Uncountable nondistributive lattice	Countable MV algebra with underlying distributive lattice[a]
Incomplete classifier	Complete classifier with equational characterization
Orthomodular logic	Infinite-valued calculus of Lukasiewicz
Unit vector in Hilbert space	Extremal positive linear normalized functional
von Neumann uniqueness theorem	Proliferation of inequivalent representations

[a]Defining in any MV algebra B, $x \leq y$, iff $x^* \oplus y = 1$, B becomes a distributive lattice. When $B = [0, 1]$, this order coincides with the natural order. However, from the order structure of B alone we cannot in general recover B uniquely. See Cignoli *et al.* (n.d.) for notable classes of exceptions.

bras, and has interesting consequences on the set of tracial states (Blackadar, 1980*b*).

By the results of Mundici (1986, §8), the AF C^*-algebra $\mathfrak{M}$ corresponding to the free MV algebra F_ω with denumerably many free generators represents a sort of "universal" infinite system, since every AF C^*-system arises from a quotient of $\mathfrak{M}$. Owing to their rich quotient structures, universal objects are important for the study of amalgamations and free products (Mundici, 1988*b*) and may serve the purpose of providing a first algebraization of composite physical systems.

In Table IV, we compare the Birkhoff–von Neumann logical analysis with our analysis of AF C^*-algebras.

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